

On the integrability of $N = 2$ supersymmetric massive theories

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Abstract

In this paper we propose a criteria to establish the integrability of $N = 2$ supersymmetric massive theories. The basic data required are the vacua and the spectrum of Bogomolnyi solitons, which can be neatly encoded in a graph (nodes=vacua and links= Bogomolnyi solitons). Integrability is then equivalent to the existence of solutions of a generalized Yang-Baxter equation which is built up from the graph (graph-Yang-Baxter equation). We solve this equation for two general types of graphs: circular and daisy, proving, in particular, the integrability of the following Landau-Ginzburg superpotentials: $A_n(t_1)$, $A_n(t_2)$, $D_n(\tau)$, $E_6(t_7)$, and $E_8(t_{16})$. For circular graphs the solution are intertwiners of the affine Hopf algebra $\tilde{U}_q(A_1^{(1)})$, while for daisy graphs the solution corresponds to a susy generalization of the Boltzmann weights of the chiral Potts model in the trigonometric regime. A chiral Potts like solution is conjectured for the more tricky case $D_n(t_2)$. The scattering theory of circular models, for instance $A_n(t_1)$ or $D_n(\tau)$, is Toda like. The physical spectrum of daisy models, as $A_n(t_2)$, $E_6(t_7)$ or $E_8(t_{16})$, is given by confined states of radial solitons. The scattering theory of the confined states is again Toda like. Bootstrap factors for the confined solitons are given by fusing the susy chiral Potts S -matrices of the elementary constituents; i.e the radial solitons of the daisy graph.

1 Introduction and Summary

It is a common believe among string practitioners that the space of two dimensional theories will become, for strings, as fundamental as space and time has been for classical physics [1]. Independently whether this idea is true or not, it is certainly a good excuse for trying to unravel the secrets of two dimensional models, in particular of those possessing $N = 2$ supersymmetry. In this paper we concentrate our efforts on the study of the integrability and soliton scattering of two dimensional $N = 2$ massive theories. The infinite symmetry of integrable models is for off-shell strings the closest we can get to on shell reparametrization invariance (conformal symmetry) which fix the vacuum solutions. We hope that the real dynamical meaning of integrability will become clear at the end of the road, surely with the arrival of a well established string field theory.

Based on non renormalization theorems, $N = 2$ massive theories are usually described by a perturbed and non degenerate Landau-Ginzburg superpotential [2, 3, 4]. In general a $N = 2$ massive theory can be described by a graph with the nodes representing the vacua and the links Bogomolnyi solitons [5]. The critical values are then interpreted as coordinates for the nodes and the soliton fermion numbers as labeling the links. An interesting question consist in deriving this graph, for susy preserving deformations of $N = 2$ superconformal field theories, directly from its chiral ring structure [3]. Notice that the graph contains information on the spectrum of the theory.

The first problem we address in this paper is the integrability of $N = 2$ massive theories characterized by the graph of vacua and Bogomolnyi solitons. We translate the question of integrability, i.e. existence of an infinite number of conserved charges, into the existence of solutions of a generalization of the standard Yang-Baxter equation based on the graph. We denote this new equation graph -Yang-Baxter (gYB) [6] . We will consider two generic types of graphs characterized by their shape : circular with all the nodes living on the same circle and daisy with all the nodes on a circle except one located at its center. In both cases we find solutions to the corresponding gYB equation provided we restrict the two-soliton scattering processes to those which are elastic and with no interchange of fermion number. The solutions for circular graphs have a neat quantum group meaning. In fact they are the intertwiners of the affine Hopf algebra $U_q(A_1^{(1)})$ extended with to extra central elements. This extension is denoted in the literature $\tilde{U}_q(A_1^{(1)})$ [7]. In this way we are able, for instance, to prove the integrability of the $D_n(\tau)$ -perturbation which was unknown until now [10] . The solution for this case only differs from the well known

$A_n(t_1)$ S -matrix [8, 9] by the eigenvalues of the extra central elements. For daisy graphs the solution, as advertised in reference [6], is given by a $N = 2$ extension of chiral Potts at the superintegrable point. This solution allows us to prove the integrability of models like $E_6(t_7)$ and $E_8(t_{16})$. All these examples, namely $D_n(\tau)$, $E_6(t_7)$ and $E_8(t_{16})$, appear with question marks in the classification given in reference [11]. The explicit proof of integrability presented in this work support the phenomenological observations of [11]. For the $D_n(t_2)$ we conjecture a solution which combines general chiral Potts and the Ising model.

The second problem is the definition of a scattering theory for solitons or in other words, to find solutions of the gYB equation satisfying crossing, unitarity and the bootstrap relations [12]. For circular graphs we don't find any surprise, the physical spectrum is the one defined by the graph with the bootstrap structure reflecting its geometry. The scattering is Toda like [13, 14, 15]. The surprise and the possible new physics come with daisy graphs. In this case there are not crossing and unitary solutions of the gYB equation. We interpret this result as indicating that the radial solitons and antisolitons don't define the physical spectrum of the theory. The bound states of radial solitons are living on a circular graph and we can expect their S -matrix, which can be obtained by fusion of the S -matrices for the radial solitons, to be a solution of the circular gYB equation. This in fact the case, moreover the solution is an intertwiner of $\tilde{U}_q(A_1^{(1)})$ with non trivial eigenvalues for the extra central elements. From this result we conclude that the physical spectrum of daisy models is a "confined" spectrum defined by the subset of nodes on the circle. This phenomena is in a certain sense dual to the one described in [16] for the soliton free regime in sine-Gordon model. There, the soliton free spectrum describes non unitary theories, while here we need to confine the spectrum in order to get a physical S -matrix satisfying bootstrap. For the Landau-Ginzburg superpotential $W = x^{k+2} - tx^2$ the previous result implies that their physical, confined spectrum is, up to fermion numbers, the same one gets for the circular Landau-Ginzburg model $W = x^{k+1} - tx$. The circular model is the most relevant perturbation of the minimal SCFT with central extension $c = \frac{3(k-1)}{k+1}$ while the one we start with is the next to the most relevant perturbation of the SCFT with central extension $c = \frac{3k}{k+2}$. An important issue is therefore the ultraviolet behaviour of daisy potentials or in other words whether the infrared "confinement" is or not associated with asymptotic freedom in the ultraviolet.

The plan of the paper is as follows. In section II we give the general definitions and explain the integrability criteria of $N = 2$ massive theories. In section III we study

circular and daisy graphs proving their integrability and applying the results to some Landau-Ginzburg models whose integrability was unclear. In section IV we consider the scattering theory and discuss the confined spectrum for daisy graphs.

2 N=2 massive theories

2.1 Definitions

Let us denote by \mathcal{A} the $N = 2$ algebra:

$$\begin{aligned}
(Q^\pm)^2 &= (\bar{Q}^\pm)^2 = \{Q^+, \bar{Q}^-\} = \{Q^-, \bar{Q}^+\} = 0, \\
\{Q^+, Q^-\} &= P, & \{\bar{Q}^+, \bar{Q}^-\} &= \bar{P}, \\
\{Q^+, \bar{Q}^+\} &= W, & \{Q^-, \bar{Q}^-\} &= \bar{W}, \\
[\mathcal{F}, Q^\pm] &= \pm Q^\pm, & [\mathcal{F}, \bar{Q}^\pm] &= \mp \bar{Q}^\pm
\end{aligned} \tag{1}$$

where Q^\pm, \bar{Q}^\pm are the susy generators, W the topological central term [17] and \mathcal{F} the fermion number. The $N = 2$ algebra is equipped with a coalgebra structure defined by the following comultiplication rules:

$$\begin{aligned}
\Delta Q^\pm &= Q^\pm \otimes \mathbf{1} + e^{\pm i\pi\mathcal{F}} \otimes Q^\pm \\
\Delta \bar{Q}^\pm &= \bar{Q}^\pm \otimes \mathbf{1} + e^{\mp i\pi\mathcal{F}} \otimes \bar{Q}^\pm \\
\Delta W &= W \otimes \mathbf{1} + \mathbf{1} \otimes W \\
\Delta P &= P \otimes \mathbf{1} + \mathbf{1} \otimes P
\end{aligned} \tag{2}$$

By a massive theory we mean one with a mass gap and non degenerate vacua. Given a couple of ordered vacua (i, j) a soliton $s_{i,j}$ is a field configuration satisfying the equation of motion and which connects the vacua i at $x = -\infty$ with the vacua j at $x = \infty$. Following the spirit of reference [5] we define a massive $N = 2$ theory by the following set of data:

D1) A graph \mathcal{G} characterized by a set of nodes $i, j, ..$ and positive integer numbers $\mu_{i,j}$ which counts the number of links between the nodes i and j .

D2) Complex coordinates w_i for the nodes and real numbers $f_{i,j}$ associated with each ordered couple of connected nodes.

The physical interpretation of these data is as follows. The nodes will represent the different vacua configurations. Each ordered couple of connected nodes is interpreted as a Bogomolnyi soliton supermultiplet, transforming under the $N = 2$ supersymmetry as follows:

$$\begin{aligned}\pi_{i,j}(\theta)(Q^-) &= \begin{pmatrix} 0 & 0 \\ \sqrt{m_{i,j}}e^{\theta/2} & 0 \end{pmatrix}, & \pi_{i,j}(\theta)(Q^+) &= \begin{pmatrix} 0 & \sqrt{m_{i,j}}e^{\theta/2} \\ 0 & 0 \end{pmatrix} \\ \pi_{i,j}(\theta)(\bar{Q}^+) &= \begin{pmatrix} 0 & 0 \\ \omega_{i,j}\sqrt{m_{i,j}}e^{-\theta/2} & 0 \end{pmatrix}, & \pi_{i,j}(\theta)(\bar{Q}^-) &= \begin{pmatrix} 0 & \omega_{i,j}^*\sqrt{m_{i,j}}e^{-\theta/2} \\ 0 & 0 \end{pmatrix} \\ \pi_{i,j}(\theta)(\mathcal{F}) &= \begin{pmatrix} f_{i,j} & 0 \\ 0 & f_{i,j} - 1 \end{pmatrix}\end{aligned}\quad (3)$$

where:

$$m_{i,j} = 2|\Delta_{i,j}|, \quad \omega_{i,j} = \frac{\Delta_{i,j}}{|\Delta_{i,j}|}, \quad \Delta_{i,j} = w_j - w_i \quad (4)$$

The parameters θ , $m_{i,j}$, $\Delta_{i,j}$ and $f_{i,j}$ are respectively the rapidity, mass, topological and fermion number of the Bogomolnyi soliton $s_{i,j}$.

The $N = 2$ massive theory admits a Landau-Ginzburg interpretation if there exist a complex superpotential W whose critical points are in correspondence with the $N = 2$ vacua and such that the complex coordinates w_i , introduced above, coincide with the critical values of W , and the fermion numbers $f_{i,j}$ are given by the index theorem formula:

$$\exp(2\pi i f_{j,k}) = \text{phase} \left(\frac{\det H(k)}{\det H(j)} \right) \quad (5)$$

with $H(j)$ the Hessian of W at the critical point j . Eq.(5) fixes the fermion number only modulo 1. In table 1 we collect the basic information concerning some simple Landau-Ginzburg superpotentials.

2.2 Integrability conditions

In this section we propose an integrability criteria for $N = 2$ massive theories. To characterize these theories we will use the geometrical data D1) and D2) defined in the previous section. As a preliminary step we first introduce some notational background. Given the $N = 2$ graph \mathcal{G} a plaquette $\left(\begin{array}{c|cc} i & l & \alpha_3 \alpha_4 \\ j & k & \alpha_1 \alpha_2 \end{array} \right)$ is defined by a set of four connected

Model	Superpotential W	Extrema	W-values	$f_{i,j}$
$A_{k+1}(t_1)$	$\frac{x^{k+2}}{k+2} - x$	$x_j = e^{\frac{2\pi ij}{k+2}}$ $j = 1, \dots, k+1$	$W_j = e^{\frac{2\pi ij}{k+1}}$	$f_{j,j+1} = \frac{1}{k+1}$
$A_{k+1}(t_2)$	$\frac{x^{k+2}}{k+2} - \frac{x^2}{2}$	$x_* = 0$ $x_j = e^{\frac{2\pi ij}{k}}$ $j = 1, \dots, k$	$W_* = 0$ $W_j = e^{\frac{4\pi ij}{k}}$	$f_{*,j} = \frac{1}{2}$ $f_{j,*} = \frac{1}{2}$
$A_{k+1}(t_k)$	$\frac{T_{k+2}(x)}{k+2}$	$x_j = 2\cos\left(\frac{\pi j}{k+2}\right)$ $j = 1, \dots, k+1$	$W_j = (-1)^j$	$f_{j,j+1} = \frac{1}{2}$
$D_{k+3}(\tau)$	$\frac{x^{k+2}}{2(k+2)} + \frac{xy^2}{2} - y$	$(x_j, y_j) = (e^{\frac{2\pi ij}{k+3}}, e^{-\frac{2\pi ij}{k+3}})$ $j = 1, \dots, k+3$	$W_j = e^{-\frac{2\pi ij}{k+3}}$	$f_{j,j+1} = \frac{2}{k+3}$
$D_{k+2}(t_2)$	$\frac{x^{k+1}}{2(k+1)} + \frac{xy^2}{2} - x$	$(x_{*1}, y_{*1}) = (0, \sqrt{2})$ $(x_{*2}, y_{*2}) = (0, -\sqrt{2})$ $(x_j, y_j) = (2^{1/k} e^{\frac{2\pi ij}{k}}, 0)$ $j = 1, \dots, k$	$W_{*1,2} = 0$ $W_j = e^{\frac{2\pi ij}{k}}$	$f_{*1,2,j} = \frac{1}{2}$ $f_{j,*1,2} = \frac{1}{2}$
$E_6(t_7)$	$\frac{x^3}{3} + \frac{y^4}{4} - xy$	$(x_*, y_*) = (0, 0)$ $(x_j, y_j) = (e^{\frac{2\pi ij}{5}}, e^{\frac{4\pi ij}{5}})$ $j = 1, \dots, 5$	$W_* = 0$ $W_j = e^{\frac{6\pi ij}{5}}$	$f_{*,j} = \frac{1}{2}$ $f_{j,*} = \frac{1}{2}$
$E_8(t_{16})$	$\frac{x^3}{3} + \frac{y^5}{5} - xy$	$(x_*, y_*) = (0, 0)$ $(x_j, y_j) = (e^{\frac{2\pi ij}{7}}, e^{\frac{4\pi ij}{7}})$ $j = 1, \dots, 7$	$W_* = 0$ $W_j = e^{\frac{6\pi ij}{7}}$	$f_{*,j} = \frac{1}{2}$ $f_{j,*} = \frac{1}{2}$

Table 1: All these examples are perturbations of ADE Landau-Ginzburg models. The superpotential $T_{k+2}(x)$ of the model A_{k+1} is the Chebishev polynomial $T_n(2 \cos \theta) = 2 \cos n\theta$.

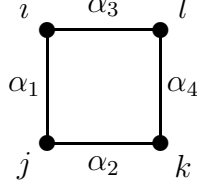


Figure 1: Plaquette

nodes i, j, k, l and four extra labels $\alpha_1, \dots, \alpha_4$ which we introduce in order to differentiate solitons interpolating the same two vacua, i.e. $\alpha_1 = 1, 2, \dots, \mu_{i,j}$, etc (see figure 1).

$N = 2$ -invariant plaquettes are those satisfying:

$$\pi_{i,j} \otimes \pi_{j,k}(\Delta(z)) = \pi_{i,l} \otimes \pi_{l,k}(\Delta(z)) \quad (6)$$

for z any central element of the $N = 2$ algebra and the irreps the ones defined in equation (3). We shall say that a $N = 2$ invariant plaquette is elastic if it satisfies the following extra condition for the masses:

$$m_{i,j} = m_{l,k} \quad , \quad m_{j,k} = m_{i,l} \quad (7)$$

Each elastic plaquette is associated with a four by four elastic S -matrix (see figure 2): $S_{m_1 m_2}^{m_3 m_4} \left(\begin{array}{c|c} i & l \\ j & k \end{array} \middle| \begin{array}{c} \alpha_3 \alpha_4 \\ \alpha_1 \alpha_2 \end{array} \right) (\theta)$, which describes the transition amplitude of the scattering process $s_{ij}(\theta_1) + s_{jk}(\theta_2) \rightarrow s_{il}(\theta_2) + s_{lk}(\theta_1)$. The $N=2$ labels m_1, \dots, m_4 take two values 0 and 1, recall eq. (3), and refer to the so called up(u) and down(d) Bogomolnyi states.

After these definitions we propose the following integrability criteria for $N = 2$ massive theories.

A $N = 2$ massive theory is integrable if the following two conditions hold:

C₁: $N = 2$ Invariance. The S -matrix associated to each elastic plaquette is an intertwiner relative to the coalgebra structure of the $N = 2$ algebra \mathcal{A} :

$$S \left(\begin{array}{c|c} i & l \\ j & k \end{array} \middle| \begin{array}{c} \alpha_3 \alpha_4 \\ \alpha_1 \alpha_2 \end{array} \right) (\theta_{12}) (\pi_{i,j}(\theta_1) \otimes \pi_{j,k}(\theta_2)) \Delta(g) = \quad (8)$$

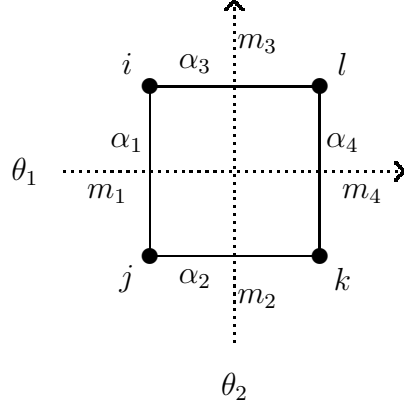


Figure 2: $N = 2$ S-matrix

$$(\pi_{i,l}(\theta_2) \otimes \pi_{l,k}(\theta_1)) \Delta(g) S \left(\begin{array}{c|c} i & l \\ j & k \end{array} \middle| \begin{array}{c} \alpha_3 \alpha_4 \\ \alpha_1 \alpha_2 \end{array} \right) (\theta_{12})$$

for g any element of the $N = 2$ algebra.

C₂: Graph-Yang-Baxter Equation (gYB):

$$\begin{aligned} & \sum_{p,m',\alpha'} S_{m_1 m_2}^{m'_1 m'_2} \left(\begin{array}{c|c} i & p \\ j & k \end{array} \middle| \begin{array}{c} \alpha'_1 \alpha'_2 \\ \alpha_1 \alpha_2 \end{array} \right) (\theta) S_{m'_2 m'_3}^{m'_3 m''_3} \left(\begin{array}{c|c} p & r \\ k & l \end{array} \middle| \begin{array}{c} \alpha'_3 \alpha''_3 \\ \alpha'_2 \alpha_3 \end{array} \right) (\theta + \theta') S_{m'_1 m'_3}^{m''_1 m''_3} \left(\begin{array}{c|c} i & s \\ p & r \end{array} \middle| \begin{array}{c} \alpha''_1 \alpha''_2 \\ \alpha'_1 \alpha'_3 \end{array} \right) (\theta') \\ &= \sum_{p,m',\alpha'} S_{m_2 m_3}^{m'_2 m'_3} \left(\begin{array}{c|c} j & p \\ k & l \end{array} \middle| \begin{array}{c} \alpha'_2 \alpha'_3 \\ \alpha_2 \alpha_3 \end{array} \right) (\theta') S_{m'_1 m'_2}^{m''_1 m''_2} \left(\begin{array}{c|c} i & s \\ j & p \end{array} \middle| \begin{array}{c} \alpha''_1 \alpha'_1 \\ \alpha_1 \alpha'_2 \end{array} \right) (\theta + \theta') S_{m'_1 m'_3}^{m''_1 m''_3} \left(\begin{array}{c|c} s & r \\ p & l \end{array} \middle| \begin{array}{c} \alpha''_2 \alpha''_3 \\ \alpha'_1 \alpha'_3 \end{array} \right) (\theta) \end{aligned} \quad (9)$$

where the sum over graph labels is restricted to elastic plaquettes (see figure 3). Observe that the gYB equation is a "fusion" of the ordinary vertex YB equation for the labels m and α and the RSOS YB equation for the labels i, j , etc..

We shall make now some comments:

1) Using the fact that different solitons interpolating the same two vacua are associated with the same $N = 2$ irrep we obtain the following factorization of the S -matrix:

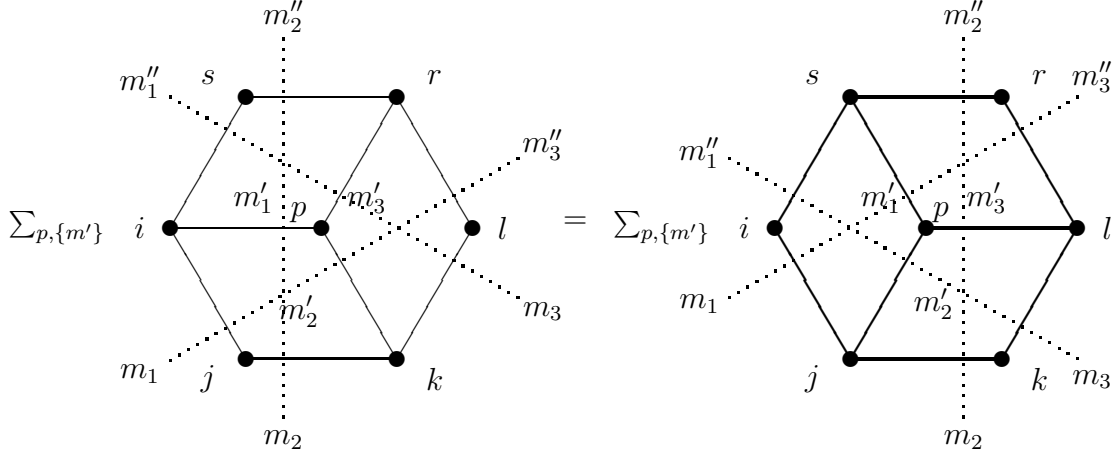


Figure 3: Graph-Yang-Baxter equation. We have not included the α 's labels in order to simplify the drawing.

$$S_{m_1 m_2}^{m_3 m_4} \left(\begin{array}{c|c} i & l \\ j & k \end{array} \middle| \begin{array}{c} \alpha_3 \alpha_4 \\ \alpha_1 \alpha_2 \end{array} \right) (\theta) = S^{(N=2)}_{m_1 m_2}^{m_3 m_4} \left(\begin{array}{c} i & l \\ j & k \end{array} \right) (\theta) S^{(N=0)} \left(\begin{array}{c|c} i & l \\ j & k \end{array} \middle| \begin{array}{c} \alpha_3 \alpha_4 \\ \alpha_1 \alpha_2 \end{array} \right) (\theta) \quad (10)$$

Notice that the requirement C_1 of $N = 2$ invariance, fix the ratios for the entries of the $N = 2$ piece of the S -matrix leaving completely undetermined the $N = 0$ part. The simplest example of the factorization (10) takes place for the *ADE* Chebishev potentials which describe the least relevant perturbation of minimal SCFT. In these models the $N = 2$ piece is independent of the vacua labels and the $N = 0$ piece is a solution to the standard *RSOS* Yang Baxter equation for the *ABF* models [18] defined by the Coxeter *ADE* graphs [9]. When the w -coordinates are degenerate, i.e. two or more nodes of the graph correspond to the same point in the w -plane, then different links of the graph will be mapped into the same one in the w -plane. All these links are associated with the same $N = 2$ irrep. Therefore the $N = 2$ part of the S matrix only depends on the w -plane graph. The model is completely characterized by the $N = 2$ part if the multiplicities $\mu_{i,j}$ are all equal to one and the w -coordinates are non degenerate, in all other cases there exist a $N = 0$ part.

2) If all multiplicities $\mu_{i,j}$ are equal to one, then the gYB equation (9) for the choices $m_1 = m_2 = m_3 = m_4 = 0$ or 1 reduces to a *RSOS* Yang-Baxter equation for the graph \mathcal{G} with the extra selection rule imposed by the elasticity condition (7). Solutions to equation

(8) and to the full fledged gYB equation (9) would be equivalent, in these cases, to the existence of a $N = 2$ supersymmetric extension of the corresponding *RSOS* statistical model.

3) The $N = 2$ massive theories can possess, in some particular cases, extra $N = 0$ symmetries that would impose additional restrictions on the $N = 0$ part of the S -matrix. The $N = 0$ symmetry, if present, will act on the multiplicity labels (α) fixing the ratios for the different entries of the $N = 0$ S -matrix. This $N = 0$ symmetry can be associated with a classical or a quantum algebra and in all cases commute with the $N = 2$ supersymmetry. CP^n sigma models and affine perturbations of Kazama-Suzuki cosets are good examples of $N = 2$ massive theories with $N = 0$ symmetries.

4) Extra integrability symmetries. A possible scenario we will often find is that of $N = 2$ massive theories where the gYB equation has not solution, unless we restrict the set of elastic plaquettes. Of course this restriction should not imply the restriction of the graph, i.e. the physical decoupling of any vacua. If this is the case we would conclude that the theory is integrable and that the extra selection rule is an integrability symmetry. In all the integrable $N = 2$ massive theories we have studied, integrability is obtained after reducing the elastic plaquettes to those with equal fermion numbers for opposite sides and therefore the fermion number appears as an "individual" conserved quantity. In principle, this extra selection rule is independent of the general Z -invariance [19] which underlines integrability.

5) The integrability of the $N = 2$ massive theories, characterized by conditions C_1 and C_2 should not be confused with the existence of a well defined scattering theory. As we will discuss in section IV in order to define a scattering theory we need to impose the additional physical requirements of unitarity, crossing and bootstrap.

6) The geometrical data D_1 and D_2 we have used to define a massive $N = 2$ theory, together with the conditions C_1 and C_2 in terms of which we characterize its integrability, can be interpreted from a mathematical point of view, as a way to define a new mathematical structure which generalizes that of quantum groups. This new mathematical object or, graph quantum group, would be defined by a Hopf algebra \mathcal{A} , a graph, and a map from links of the graph into irreps of the Hopf algebra in such a way that for any plaquette of the graph there exist an intertwiner satisfying conditions C_1 and C_2 . The main difference with respect to ordinary quantum groups is that now the intertwiner establish an equivalence between irreps which are not necessarily related by a permutation. Therefore these intertwiners cannot follow from a universal R -matrix.

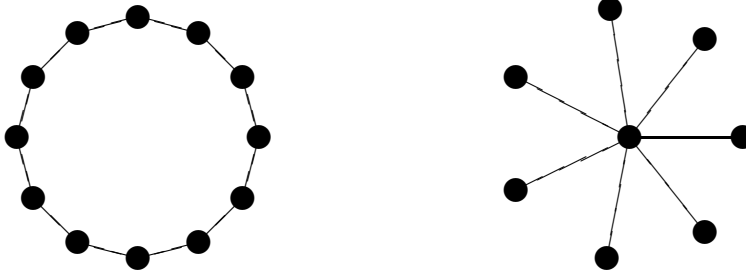


Figure 4: Circular and daisy graphs

3 Partial Classification of Integrable $N = 2$ Massive Theories

In this section we present a partial classification of integrable $N = 2$ theories, mostly based on the geometry of the graph. We consider two basic types of geometries, which seem to be the basic building blocks of most of the known soliton polytopes [15], namely circular graphs with all the nodes on the same circle and daisy graphs with all nodes on the same circle except one which is located at the center (see fig. 4).

3.1 Circular graphs

Let be a generic circular graph with n uniformly distributed nodes $j = 1, 2, \dots, n$ and links $(j, j+r)$ connecting arbitrary couples of different nodes. The simplest links $(j, j+1)$ and $(j, j-1)$ will be interpreted as the fundamental soliton and antisoliton. To fix the model we give the fermion number f of the fundamental solitons and the w -coordinates of the nodes. Without loss of generality we can take as w -coordinates the circular angles as described in figure 5. The fermion number of non fundamental solitons $(j, j+r)$ is determined by the comultiplication rule of the fermion number. Notice that for circular graphs the fermion number of the antisoliton $(j, j-1)$ coincides, modulo 1, with the one of the composite soliton $(j, j+n-1)$. A generic elastic plaquette $\begin{pmatrix} i & l \\ j & k \end{pmatrix}$ (for simplicity we consider all multiplicities equal to one) describes an elastic scattering process where the "in" state is given by the ordered set of vacua (i, j, k) and the "out" state by the set

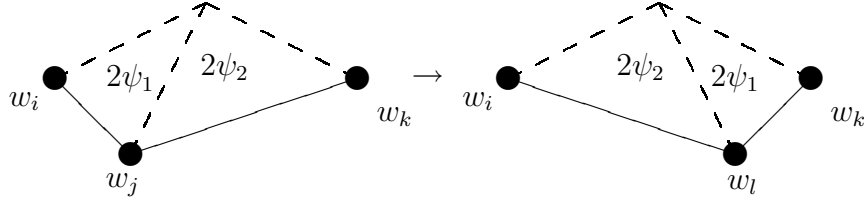


Figure 5: Picture on the w plane of an elastic scattering process

(i, l, k) . The w -coordinates of the "in" state are the two circular angles $2\psi_1, 2\psi_2$ determined by the two incoming solitons. By elasticity the angles characterizing the "out" state will be $2\psi_2, 2\psi_1$ (see figure 5).

Notice that the condition of elasticity allows plaquettes of the type $\begin{pmatrix} i & j \\ j & i \end{pmatrix}$ which correspond to the same in and out state with the angles given by ψ_1 and $-\psi_1$. These type of plaquettes can be avoided if we impose, in addition to elasticity, the extra condition:

$$f_{i,j} = f_{l,k} \quad , \quad f_{j,k} = f_{i,l} \quad (11)$$

i.e. equal fermion number for opposite sides of the plaquette. As it would be clear from the explicit computations below, integrability for circular graphs, i.e. solutions to the gYB equation, will require to impose condition (11) on the fermion numbers. This is the type of integrability symmetries we have discussed in the previous section. In table 2 we show the ratios between the entries of the $N = 2$ S -matrix for the scattering process described in figure 5. They can be deduced solving the intertwiner condition (8). The fermion numbers of the incoming solitons are denoted by f_1, f_2 .

Once we have obtained the $N = 2$ ratios we move into the question of integrability, i.e. to solve the gYB equation. Restricting ourselves to elastic plaquettes satisfying (11), the gYB equation reduces to the standard vertex YB equation where each line is associated with a rapidity θ (elasticity), "angle" ψ and fermion number f (condition (11)). It is not difficult to check that the S -matrix given in table 2 does satisfy this vertex Yang-Baxter equation (there is of course an overall factor multiplying each S -matrix which does not affect the YB equation).

The solution given in table 2 have a very interesting quantum group meaning, in fact

m_3 m_1	m_4 m_2	$S_{m_1 m_2}^{m_3 m_4}(\psi_1, \psi_2) / S_{0 0}^{0 0}(\psi_1, \psi_2)$
1 1	1 1	$-e^{i\pi(f_1-f_2)} e^{i(\psi_2-\psi_1)} \frac{\sinh(\frac{\theta}{2}-i\frac{\psi_1+\psi_2}{2})}{\sinh(\frac{\theta}{2}+i\frac{\psi_1+\psi_2}{2})}$
1 0	0 1	$e^{i\pi f_1} e^{-i\psi_1} \frac{\sinh(\frac{\theta}{2}-i\frac{\psi_1-\psi_2}{2})}{\sinh(\frac{\theta}{2}+i\frac{\psi_1+\psi_2}{2})}$
0 1	1 0	$e^{-i\pi f_2} e^{i\psi_2} \frac{\sinh(\frac{\theta}{2}+i\frac{\psi_1-\psi_2}{2})}{\sinh(\frac{\theta}{2}+i\frac{\psi_1+\psi_2}{2})}$
0 0	1 1	$i e^{i\pi(f_1-f_2)} e^{-i\frac{\psi_1-\psi_2}{2}} \frac{(\sin\psi_1 \sin\psi_2)^{1/2}}{\sinh(\frac{\theta}{2}+i\frac{\psi_1+\psi_2}{2})}$
1 1	0 0	$i e^{-i\frac{\psi_1-\psi_2}{2}} \frac{(\sin\psi_1 \sin\psi_2)^{1/2}}{\sinh(\frac{\theta}{2}+i\frac{\psi_1+\psi_2}{2})}$

Table 2: Relations imposed by the N=2 intertwiner condition on "circular" S -matrices.

it is the quantum R -matrix intertwiner for the Hopf algebra $\tilde{U}_q(A_1^{(1)})$ with deformation parameter $q^4 = 1$. This Hopf algebra, which was introduced in reference [7], is very close to the Hopf algebra $U_q(A_1^{(1)})$. The only changes are the addition of two new central elements $Z_i (i = 0, 1)$ which modify the usual comultiplication rules of the rest of the elements of $U_q(A_1^{(1)})$ as follows :

$$\begin{aligned}
\Delta E_i &= E_i \otimes \mathbf{1} + Z_i K_i \otimes E_i \\
\Delta F_i &= F_i \otimes K_i^{-1} + Z_i^{-1} \otimes F_i \\
\Delta K_i &= K_i \otimes K_i \\
\Delta Z_i &= Z_i \otimes Z_i
\end{aligned} \tag{12}$$

We shall be interested in the so called nilpotent representations [20] of the $\tilde{U}_q(A_1^{(1)})$ algebra which are labelled, in addition to the rapidity θ , by a pair $\xi = (\lambda, z)$ of non zero complex numbers. They are given by:

$$\pi_{\lambda, z}(\theta)(E_0) = d(\lambda) \begin{pmatrix} 0 & 0 \\ e^{\theta/2} & 0 \end{pmatrix}, \quad \pi_{\lambda, z}(\theta)(E_1) = d(\lambda) \begin{pmatrix} 0 & e^{\theta/2} \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
\pi_{\lambda,z}(\theta)(F_0) &= d(\lambda) \begin{pmatrix} 0 & e^{-\theta/2} \\ 0 & 0 \end{pmatrix}, & \pi_{\lambda,z}(\theta)(F_1) &= \begin{pmatrix} 0 & 0 \\ e^{-\theta/2} & 0 \end{pmatrix} \\
\pi_{\lambda,z}(\theta)(K_0) &= \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}, & \pi_{\lambda,z}(\theta)(K_1) &= \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \\
\pi_{\lambda,z}(\theta)(Z_0) &= \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}, & \pi_{\lambda,z}(\theta)(Z_1) &= \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix},
\end{aligned} \tag{13}$$

where $d(\lambda) = \left(\frac{\lambda - \lambda^{-1}}{2i}\right)^{1/2}$.

The intertwiner matrix $R(\lambda_1, z_1; \lambda_2, z_2; \theta)$ of the tensor product of two nilpotents irreps can be computed using equations (12, 13) and reads:

$$\begin{aligned}
R_{00}^{00} &= \frac{1}{2} \left(\frac{z_2}{z_1}\right)^{1/2} \left[e^{\theta/2} (\lambda_1 \lambda_2)^{1/2} - e^{-\theta/2} (\lambda_1 \lambda_2)^{-1/2} \right] \\
R_{11}^{11} &= \frac{1}{2} \left(\frac{z_1}{z_2}\right)^{1/2} \left[e^{-\theta/2} (\lambda_1 \lambda_2)^{1/2} - e^{\theta/2} (\lambda_1 \lambda_2)^{-1/2} \right] \\
R_{01}^{00} &= \frac{1}{2} (z_1 z_2)^{1/2} \left[e^{\theta/2} \left(\frac{\lambda_2}{\lambda_1}\right)^{1/2} - e^{-\theta/2} \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \right] \\
R_{10}^{00} &= \frac{1}{2} (z_1 z_2)^{-1/2} \left[e^{\theta/2} \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} - e^{-\theta/2} \left(\frac{\lambda_2}{\lambda_1}\right)^{1/2} \right] \\
R_{01}^{01} &= \frac{1}{2} \left(\frac{z_1 \lambda_1}{z_2 \lambda_2}\right)^{1/2} \left[(\lambda_1 - \lambda_1^{-1}) (\lambda_2 - \lambda_2^{-1}) \right]^{1/2} \\
R_{10}^{01} &= \frac{1}{2} \left(\frac{z_2 \lambda_2}{z_1 \lambda_1}\right)^{1/2} \left[(\lambda_1 - \lambda_1^{-1}) (\lambda_2 - \lambda_2^{-1}) \right]^{1/2}
\end{aligned} \tag{14}$$

The restriction to elastic plaquettes satisfying the fermion number condition (11) allow us to associate with these plaquettes two irreps $\xi_1 = (\lambda_1, z_1)$ and $\xi_2 = (\lambda_2, z_2)$ of $\tilde{U}_q(A_1^{(1)})$ where λ and z are given, in terms of the $N = 2$ data, by the following relations:

$$\lambda = e^{i\psi}, \quad z = e^{i\pi f} e^{-i\psi} \tag{15}$$

Using these identifications it is easy to see that the $N = 2$ S -matrix of table 2 coincides, up to an overall factor, with the R -matrix (14).

Taking into account eq.(15) we deduce that the well known relation between the $N = 2$ algebra with the quantum Hopf algebra $U_q(A_1^{(1)})$ given by [21, 22] :

$$\begin{aligned}
Q^+ &= E_1, & \bar{Q}^+ &= F_1 K_1 \\
Q^- &= E_0, & \bar{Q}^- &= K_0 F_0
\end{aligned} \tag{16}$$

is completed in the $\tilde{U}_q(A_1^{(1)})$ algebra by the relation:

$$e^{i\pi\mathcal{F}} = Z_1 K_1 \quad (17)$$

In summary we obtain the following general result:

All $N = 2$ massive theories with all critical points uniformly distributed on a circle are integrable. Moreover the $N = 2$ piece of the scattering S -matrix is given by the quantum intertwiner of $\tilde{U}_q(A_1^{(1)})$ for the irreps defined by equation (15).

Next we present some Landau-Ginzburg examples:

- 1) $A_n(t_1) : \lambda_a = e^{i\pi a/n}, z_a = 1 \ (a = 1, \dots, n-1)$
- 2) $D_n(\tau) : \lambda_a = e^{i\pi a/n}, z_a = e^{-3i\pi a/n} \ (a = 1, \dots, n-1).$
- 3) $A_n(t_{n-1}) : \lambda = e^{i\pi/2}, z = 1$

In cases 1) and 2) the full S -matrices are given by the quantum intertwiners of $\tilde{U}_q(A_1^{(1)})$. The only difference between these two cases resides in the values of the central elements z 's which reflect the differences in fermion numbers (see eq. (15)).

The case 3) corresponds to the Chebishev potential for the A models. In this case the graph defined by the vacua coincides with the Coxeter diagram of type A . The w -coordinates are degenerate and the image of the graph in the w -plane corresponds to the limit case of the circular graph defined by two points linked by a line. The $N = 2$ piece of the S -matrix is simply the R -matrix of the quantum group $U_q(A_1^{(1)})$ for the spin $\frac{1}{2}$ representation. This is also the S -matrix of the sine-Gordon model at coupling $\beta^2 = \frac{2}{3}8\pi$. The $N=0$ piece is given by the ABF solution for type A $RSOS$ models.

3.2 Daisy Graphs

Now we consider uniform daisy graphs of the type depicted in figure 4, with k nodes on the circle. We will take as fundamental the radial soliton and antisolitons $(a, *)$, $(*, a)$ assigning to both the same fermion number f which in all our examples turns out to be equal to $\frac{1}{2}$ (see table 1). As w -coordinates we shall choose zero for the central node $*$ and $w_a = e^{4\pi i a/k} (a = 1, \dots, k)$ (this choice is motivated by the study of the case $A_{k+1}(t_2)$, see table 1). For these graphs there exist two different types of elastic plaquettes namely: $\begin{pmatrix} a & * \\ * & b \end{pmatrix}$ and $\begin{pmatrix} * & b \\ a & * \end{pmatrix}$. As we showed for the circular case, the $N = 2$ invariance fixes the ratios between the various entries of the daisy S -matrices. In table 3 we collect our results for the two types of plaquettes.

$\begin{smallmatrix} k \ell \\ i j \end{smallmatrix}$	$S_{ij}^{k\ell} \begin{pmatrix} a * \\ * b \end{pmatrix} / S_{00}^{00} \begin{pmatrix} a * \\ * b \end{pmatrix}$	$S_{ij}^{k\ell} \begin{pmatrix} * b \\ a * \end{pmatrix} / S_{00}^{00} \begin{pmatrix} * b \\ a * \end{pmatrix}$
$\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}$	$\frac{\cosh(\frac{\theta}{2} + \frac{2\pi i n}{k})}{\cosh(\frac{\theta}{2} - \frac{2\pi i n}{k})}$	$\exp(-\frac{4\pi i n}{k})$
$\begin{smallmatrix} 10 \\ 01 \end{smallmatrix}$	$-\frac{\mathrm{i}\exp(2\pi i n/k)\sinh(\frac{\theta}{2})}{\cosh(\frac{\theta}{2} - \frac{2\pi i n}{k})}$	$-\frac{\mathrm{i}\exp(-2\pi i n/k)\sinh(\frac{\theta}{2} - \frac{2\pi i n}{k})}{\cosh\frac{\theta}{2}}$
$\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}$	$-\frac{\mathrm{i}\exp(-2\pi i n/k)\sinh(\frac{\theta}{2})}{\cosh(\frac{\theta}{2} - \frac{2\pi i n}{k})}$	$-\frac{\mathrm{i}\exp(-2\pi i n/k)\sinh(\frac{\theta}{2} + \frac{2\pi i n}{k})}{\cosh\frac{\theta}{2}}$
$\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}$	$\frac{\cos(\frac{2\pi n}{k})}{\cosh(\frac{\theta}{2} - \frac{2\pi i n}{k})}$	$\frac{\exp(-2\pi i n/k)\cos(\frac{2\pi n}{k})}{\cosh\frac{\theta}{2}}$
$\begin{smallmatrix} 01 \\ 01 \end{smallmatrix}$	$\frac{\cos(\frac{2\pi n}{k})}{\cosh(\frac{\theta}{2} - \frac{2\pi i n}{k})}$	$\frac{\exp(-2\pi i n/k)\cos(\frac{2\pi n}{k})}{\cosh\frac{\theta}{2}}$

Table 3: Relations imposed by the N=2 intertwiner condition

Summing over all elastic plaquettes the gYB equation for daisy graphs is given by:

$$\begin{aligned}
R(\theta_1, \theta_2, \theta_3) & \sum_{m'_1, m'_2, m'_3} S_{m_1 m_2}^{m'_1 m'_2} \begin{pmatrix} a * \\ * b \end{pmatrix} (\theta_{12}) S_{m'_2 m'_3}^{m'_3 m''_3} \begin{pmatrix} * c \\ b * \end{pmatrix} (\theta_{13}) S_{m'_1 m'_3}^{m''_1 m''_2} \begin{pmatrix} a * \\ * c \end{pmatrix} (\theta_{23}) \\
& = \sum_d \sum_{m'_1, m'_2, m'_3} S_{m'_2 m'_3}^{m'_2 m'_3} \begin{pmatrix} * d \\ b * \end{pmatrix} (\theta_{23}) S_{m'_1 m'_2}^{m''_1 m'_1} \begin{pmatrix} a * \\ * d \end{pmatrix} (\theta_{13}) S_{m'_1 m'_3}^{m''_2 m'_3} \begin{pmatrix} * c \\ d * \end{pmatrix} (\theta_{12}) \quad (18)
\end{aligned}$$

The graphical meaning of this equation is given in figure 6.

Before entering into the description of the solution to equation (18), we would like to make some preliminary comments on its structure and physical meaning:

1) The first relevant thing to be noticed is that due to the geometry of daisy graphs, the gYB equation is asymmetric, appearing the sum over graph labels only in one of the terms of the equation. This is a well known phenomena in the *RSOS* description of chiral Potts models [23, 24]. In fact if we consider equation (18) for the susy labels $m_1, m_2, m_3, m_4 = 0, \text{or } 1$, the two resulting equations are exactly the RSOS version of the YB equation for a Z_k chiral Potts model [24].

2) In equation (18) we have included an extra factor R . The reason for including this factor is certainly inspired by the form of the star triangle relation of the chiral Potts

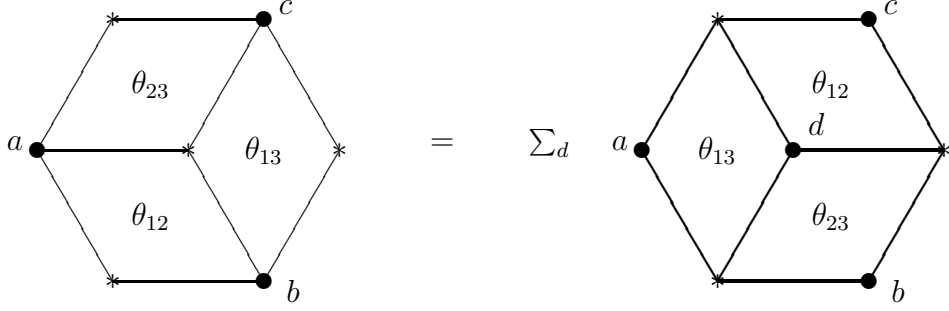


Figure 6: Star-triangular relation

model. As it happens overthere this factor does not affect the integrability of the theory, which is certainly the topic of this section. From a physical point of view this factor should be put equal to one in order to interpret the solutions to (18) as the physical scattering S matrices. We shall follow instead the strategy of leaving this factor undetermined in the general discussion of integrability, while fixing it in the scattering theory section. This will require to discover the physical spectrum of asymptotic particles.

After these general remarks we are ready to give the explicit solution. Using the $N = 2$ relations of table 3, the solution is completely characterized by the values of two entries. After a lengthly computation we find the following solution:

$$\begin{aligned}
S_{00}^{00} \begin{pmatrix} a * \\ * b \end{pmatrix} (\theta) &= S(\theta) \prod_{r=0}^{a-b-1} \frac{\cosh(\frac{\theta}{2} + \frac{2\pi i r}{k})}{\cosh(\frac{\theta}{2} - \frac{2\pi i r}{k})} \\
S_{00}^{00} \begin{pmatrix} * b \\ a * \end{pmatrix} (\theta) &= \bar{S}(\theta) e^{2\pi i(a-b)/k} \prod_{r=0}^{a-b-1} \frac{\sinh(\frac{\theta}{2} - \frac{2\pi i r}{k})}{\sinh(\frac{\theta}{2} + \frac{2\pi i(r+1)}{k})}
\end{aligned} \tag{19}$$

This is a solution to equation (18) with the factor R given in term of the two undetermined functions $S(\theta)$ and $\bar{S}(\theta)$ as follows:

$$\begin{aligned}
R(\theta_1, \theta_2, \theta_3) &= \frac{f(\theta_{12})f(\theta_{23})}{f(\theta_{13})} \\
f(\theta) &= \frac{\bar{S}(\theta)}{S(\theta)} \sum_{m=1}^k e^{2\pi i m/k} \prod_{r=0}^{m-1} \frac{\sinh(\frac{\theta}{2} - \frac{2\pi i r}{k})}{\sinh(\frac{\theta}{2} + \frac{2\pi i(r+1)}{k})}
\end{aligned} \tag{20}$$

The last formula for the factor $f(\theta)$ can be simplified using the identity:

$$\sum_{m=1}^k e^{2\pi i m/k} \prod_{r=0}^{m-1} \frac{\sinh(\frac{\theta}{2} - \frac{2\pi i r}{k})}{\sinh(\frac{\theta}{2} + \frac{2\pi i(r+1)}{k})} = \sqrt{k} e^{3\pi i(k-1)/4} \prod_{j=1}^{(k-1)/2} \frac{\cosh(\frac{\theta}{2} - \frac{2\pi i j}{k})}{\sinh(\frac{\theta}{2} + \frac{2\pi i j}{k})} \quad (21)$$

The previous solution (19) have an intrinsic meaning from the point of view of chiral Potts. In fact the entries S_{00}^{00} and S_{11}^{11} are two different trigonometric solution of the chiral Potts model characterized by a parameter $\omega = e^{4\pi i/k}$, "moduli" $k' = 1$ and "chiral angles":

$$\phi = \frac{\pi}{2}(k \mp 2), \quad \bar{\phi} = \frac{\pi}{2}(k \pm 2) \quad \text{for } S_{00}^{00} \text{ } (S_{11}^{11}) \quad (22)$$

Eq (22) suggest some kind of relation with the so called superintegrable chiral Potts model which corresponds to the values $\phi = \bar{\phi} = \frac{\pi}{2}$ [25]. This model has received much attention in the past due to its peculiar properties which singularize it among the more general class of chiral Potts models [26]. The general daisy graph solution we have described above can be used to analyze the integrability of some Landau-Ginzburg potentials. Next we describe five different models all of them associated with daisy graphs (see table 1 for notations).

1) $A_{k+1}(t_2)$, $k=\text{odd}$. This model corresponds exactly to the solution we have just described.

2) $A_{k+1}(t_2)$, $k=\text{even}$. In this case we find the phenomena of degenerate w -coordinates as can be seen from the fact that the k critical points $x_j = e^{2\pi i j/k}$ ($j = 1, \dots, k$) are mapped onto $k/2$ distinct points $W_j = e^{4\pi i j/k}$ in the w -plane. We can differentiate two different subcases depending on the parity of $k/2$. If $k/2$ is even there are three collinear points in the w -plane, while if $k/2$ is odd this situation does not occur. In the later case the $N = 2$ part of the S -matrices is again given by eqs.(19), which must be supplemented with a $N = 0$ piece to take care of the double degeneracy of the vacua in the w -plane. The case of collinear vacua is more subtle since, as can be seen from table 3, one gets singular values for some entries of the S -matrices. These singularities are due to the fact that whenever we have three collinear points a two soliton state is undistinguishable from a single soliton state (see reference [5]).

3) $E_6(t_7)$. The solution of this model is the same as the one for $A_6(t_2)$.

4) $E_8(t_{16})$. The solution of this model is the same as the one for $A_8(t_2)$.

5) $D_{k+2}(t_2)$. For simplicity we consider the $k=\text{odd}$ case. From table 1 we observe that the graph is actually three dimensional with k nodes on the x plane, all on the same

circle and two extra nodes $(*_\alpha, \alpha = 1, 2)$ on the y line. Inspired by the solution to the daisy models we can conjecture the following factorization of the S -matrices:

$$S_{m_1 m_2}^{m_3 m_4} \begin{pmatrix} a & *_\beta \\ *_\alpha & b \end{pmatrix} (\theta) = S_{m_1 m_2}^{m_3 m_4} \begin{pmatrix} a & * \\ * & b \end{pmatrix} (\theta) S \begin{pmatrix} \diamond \beta \\ \alpha \diamond \end{pmatrix} (\theta) \quad (23)$$

and similarly for the other type of plaquettes. In this factorization the $N = 2$ part is the solution for the daisy graph obtained by projecting on the w -plane. The $N = 0$ part is the standard solution for the Ising model at criticality where now the lattice variables are identified with the labels $\alpha, \beta = 1, 2$.

4 Scattering Theory

The question of integrability of a massive $N = 2$ theory and its reduction to a scattering theory satisfying bootstrap and factorization are two different but related questions. In the study of integrability we start with a graph and formally assume that all links of the graph correspond to real asymptotic particles. In the spirit of this assumption we solve the gYB equation and interpret the solutions as scattering S matrices. To promote these solutions, whose existence already implies an infinite number of conserved charges, to a real scattering S matrix, requires to impose unitarity, crossing and bootstrap. Only after fulfilling these physical requirements we can be sure that the $N = 2$ massive theory is equivalent to a scattering theory satisfying Zamolodchikov's axiomatics [12] and that the links of the graph actually represent the real asymptotic particles.

In this section we will define a closed, in the bootstrap sense, scattering theory for the two general types of $N = 2$ massive theories we have described until now, namely those associated with circular and daisy graphs. The results we find are the following. For circular graphs the scattering theory is obtained by solving bootstrap equations which are analogous to the ones describing Toda type theories [13, 14, 15]. The case of daisy graphs is physically more interesting, as can be already expected from the chiral Potts solution. A consistent scattering theory can be defined only after reducing the physical spectrum to composite solitons obtained as soliton-antisoliton bound states. The scattering of these composite solitons is derived from the chiral Potts solution for radial soliton- antisoliton scattering by a "fusion" procedure. The resulting scattering theory is again Toda like of the same type that for the $D_n(\tau)$ model, in the sense that the central elements Z_1 of

$\tilde{U}_q(A_1^{(1)})$ take non trivial values. After these introductory remarks we pass to present our results.

4.1 Circular Scattering: Toda like spectrum

In the circular case the S -matrix is given by the R -matrix (14) up to an overall factor Z which depends on the irreps λ_1, λ_2 which one fixes imposing unitarity, crossing and bootstrap [9]. Let us suppose that there are n vacua equally space on the same circle $j = 1, \dots, n$. The value of λ for the soliton $(j, j+r)$ is given by $\lambda = e^{i\pi r/n}$, while we shall leave the value of z undetermined. Then the S -matrix describing the scattering of the soliton $(j, j+r_1)$ with rapidity θ_1 and the soliton $(j+r_1, j+r_1+r_2)$ with rapidity θ_2 is given by:

$$S \left(\begin{array}{cc} j & j+r_2 \\ j+r_1 & j+r_1+r_2 \end{array} \right) (\theta_{12}) = Z_{r_1, r_2}(\theta_{12}) R(\lambda_1 = e^{i\pi r_1/n}, z_{r_1}; \lambda_2 = e^{i\pi r_2/n}, z_{r_2}, \theta_{12}) \quad (24)$$

Unitarity implies the equation:

$$S \left(\begin{array}{cc} j & j+r_2 \\ j+r_1 & j+r_1+r_2 \end{array} \right) (\theta) S \left(\begin{array}{cc} j & j+r_1 \\ j+r_2 & j+r_1+r_2 \end{array} \right) (-\theta) = \mathbf{1} \quad (25)$$

and crossing:

$$Z_{r_1, r_2}(\theta) = Z_{n-r_2, r_1}(i\pi - \theta) \quad (26)$$

The analysis of the bootstrap properties of this model leads finally to the following expresion of $Z_{1,1}$ (see reference [9] for details):

$$\begin{aligned} Z_{1,1}(\theta) &= \frac{1}{\sinh(\frac{\theta}{2} - \frac{i\pi}{n})} \prod_{j=1}^{\infty} \frac{\Gamma^2(-\theta/2\pi i + j) \Gamma(\theta/2\pi i + j + 1/n) \Gamma(\theta/2\pi i + j - 1/n)}{\Gamma^2(\theta/2\pi i + j) \Gamma(-\theta/2\pi i + j + 1/n) \Gamma(-\theta/2\pi i + j - 1/n)} \\ &= \frac{1}{\sinh(\frac{\theta}{2} - \frac{i\pi}{n})} \exp \left(2i \int_0^\infty \frac{dt}{t} \sin t \theta \frac{\sinh^2 \pi t/n}{\sinh^2 \pi t} \right) \end{aligned} \quad (27)$$

We shall return to this factor in the next subsection.

4.2 Daisy Scattering: Confinement like spectrum

The solution (19) to the gYB equation for daisy graphs was fixed up to two undetermined functions $S(\theta)$ and $\bar{S}(\theta)$. We can use this freedom in order to define an unitary and crossing symmetric S-matrix satisfying at the same time the gYB equation (18) with the factor R set equal to one. It is easy to check that there is not solution to all these conditions. Let us show why this happens in more detail. Crossing symmetry is guaranteed if:

$$S_{m_1 m_2}^{m_3 m_4} \begin{pmatrix} * & b \\ a & * \end{pmatrix} (\theta) = S_{m_2 \bar{m}_4}^{\bar{m}_1 m_3} \begin{pmatrix} a & * \\ * & b \end{pmatrix} (i\pi - \theta) \quad (28)$$

where $\bar{m} = 1, 0$ for $m = 0, 1$. This implies in turn the following relation between $S(\theta)$ and $\bar{S}(\theta)$:

$$\bar{S}(\theta) = i \cotanh\left(\frac{\theta}{2}\right) S(i\pi - \theta) \quad (29)$$

Introducing eq.(29) into (20) and using the relation (21) one deduces:

$$f(\theta) f(i\pi - \theta) = k \quad (30)$$

which already implies that the factor R cannot be set equal to one in the gYB equation. Moreover, the unitarity conditions:

$$\begin{aligned} S \begin{pmatrix} a & * \\ * & b \end{pmatrix} (\theta) S \begin{pmatrix} a & * \\ * & b \end{pmatrix} (-\theta) &= \mathbf{1} \\ \sum_b S \begin{pmatrix} * & a \\ b & * \end{pmatrix} (\theta) S \begin{pmatrix} * & b \\ c & * \end{pmatrix} (-\theta) &= \delta_{a,c} \mathbf{1} \end{aligned} \quad (31)$$

are incompatible with crossing. The best that can be done is to find solutions which satisfy unitarity, but violate crossing by a constant and satisfy the gYB equation with the factor R also a constant. Notice that the violation of crossing and the existence of the anomalous factor R in the factorization equations, are two related problems. In fact by crossing we relate the two types of elastic plaquettes, and the mismatch in the crossing relation show up in the factor R . The physical origin of these problems can be partially understood as a consequence of the special symmetry properties of the chiral Potts Boltzmann weights. In fact the solution (19) is neither P nor T invariant, however it satisfies the most general requirement of PCT invariance:

$a - b$	0	1	2
$S_{0\ 0}^{0\ 0} \begin{pmatrix} a\ * \\ * b \end{pmatrix}$	$S(\theta)$	$S(\theta)$	$S(\theta) \frac{\cosh(\frac{\theta}{2} + \frac{2\pi i}{3})}{\cosh(\frac{\theta}{2} - \frac{2\pi i}{3})}$
$S_{1\ 1}^{1\ 1} \begin{pmatrix} a\ * \\ * b \end{pmatrix}$	$S(\theta)$	$S(\theta) \frac{\cosh(\frac{\theta}{2} + \frac{2\pi i}{3})}{\cosh(\frac{\theta}{2} - \frac{2\pi i}{3})}$	$S(\theta)$
$S_{0\ 0}^{0\ 0} \begin{pmatrix} * b \\ a\ * \end{pmatrix}$	$\bar{S}(\theta)$	$\bar{S}(\theta) \frac{e^{2\pi i/3} \sinh \frac{\theta}{2}}{\sinh(\frac{\theta}{2} + \frac{2\pi i}{3})}$	$\bar{S}(\theta) \frac{e^{-2\pi i/3} \sinh \frac{\theta}{2}}{\sinh(\frac{\theta}{2} + \frac{2\pi i}{3})}$
$S_{1\ 1}^{1\ 1} \begin{pmatrix} * b \\ a\ * \end{pmatrix}$	$\bar{S}(\theta)$	$\bar{S}(\theta) \frac{e^{-2\pi i/3} \sinh \frac{\theta}{2}}{\sinh(\frac{\theta}{2} + \frac{2\pi i}{3})}$	$\bar{S}(\theta) \frac{e^{2\pi i/3} \sinh \frac{\theta}{2}}{\sinh(\frac{\theta}{2} + \frac{2\pi i}{3})}$

Table 4: S -matrix elements for $k = 3$

$$S_{m_1\ m_2}^{m_3\ m_4} \begin{pmatrix} a\ * \\ * b \end{pmatrix} (\theta) = S_{\tilde{m}_4\ \tilde{m}_3}^{\tilde{m}_2\ \tilde{m}_1} \begin{pmatrix} b\ * \\ * a \end{pmatrix} (i\pi - \theta) \quad (32)$$

We shall follow a different strategy for associating to daisy graphs a well defined scattering theory without the anomalies described above. First of all we reduce the spectrum to composite solitons (a, b) defined by a pair of radial soliton $(a, *)$ and antisoliton $(*, b)$, then by means of a fusion procedure we compute their S matrix and finally we use the freedom of the unknown functions $S(\theta)$ and $\bar{S}(\theta)$ to solve the bootstrap equations. To simplify matters we shall work out explicitly the simplest non trivial case $k=3$. We proceed in two steps: i) fusion and ii) bootstrap.

i) Fusion:

In order to discuss the case $k = 3$ with some detail it is quite convenient to construct the table 4 which gives the relevant S -matrix elements between the solitons $(a, *)$ and $(*, b)$.

Looking at the soliton-antisoliton process $s_{a,*}(\theta_1) + s_{*,b}(\theta_2)$ one obtains the Bogomolnyi solitons $(a \pm 1, a)$ for the value $\theta_{12} = \frac{i\pi}{3}$. If moreover the function $S(\theta)$ were non vanishing at $\theta = \frac{i\pi}{3}$ then these states would simply be bound states of the radial ones. We shall see at the end of this section what is the fate of $S(\theta = \frac{i\pi}{3})$. In any case and for our purposes we shall consider the states $(a \pm 1, a)(\theta)$, which have a mass $m_{a \pm 1, a} = 2 m_{*, a} \cos(\mu/2)$ ($\mu = \frac{\pi}{3}$), as made of the pairs $s_{a \pm 1, *}(\theta + i\mu/2) \otimes s_{*, a}(\theta - i\mu/2)$. This will allow us to compute the

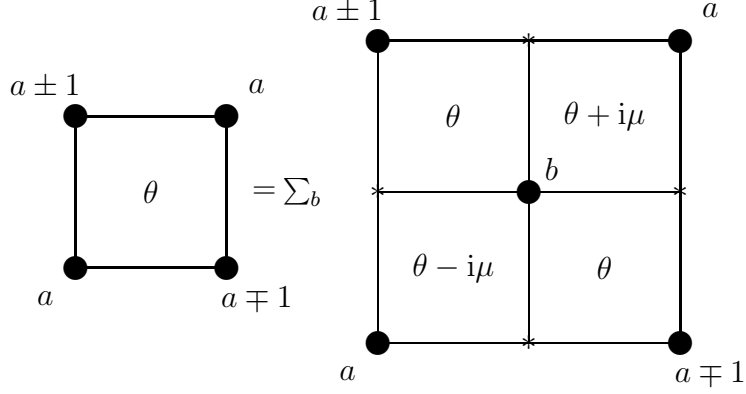


Figure 7: Fusion of daisy S -matrices to produce circular ones.

S -matrix associated with the plaquettes: $\begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix}$ by the fusion of four elementary plaquettes (see figure 7):

$$S \begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix} (\theta) = \sum_b S \begin{pmatrix} a \pm 1 & * \\ * & b \end{pmatrix} (\theta) S \begin{pmatrix} b & * \\ * & a \mp 1 \end{pmatrix} (\theta) S \begin{pmatrix} * & b \\ a & * \end{pmatrix} (\theta - i\mu) S \begin{pmatrix} * & a \\ b & * \end{pmatrix} (\theta + i\mu) \quad (33)$$

In these equations the $N = 2$ labels have been skipped for simplicity (they are all 1's or 0's for the upper or lower choices of signs). Using now table 4 we obtain :

$$S \begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix} (\theta) = -3i \frac{\sinh \frac{\theta}{2} \sinh \left(\frac{\theta}{2} + i\frac{\mu}{2} \right) \sinh \left(\frac{\theta}{2} - i\mu \right)}{\cosh \frac{\theta}{2} \cosh^2 \left(\frac{\theta}{2} - 2i\mu \right)} S(\theta)^2 \bar{S}(\theta + i\mu) \bar{S}(\theta - i\mu) \quad (34)$$

Considering the remaining $N = 2$ entries of this S - matrix one recovers the intertwiner R matrix of $\tilde{U}_q(A_1^{(1)})$ Hopf algebra:

$$S \begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix} (\theta) = 3i \frac{\sinh \frac{\theta}{2} \sinh \left(\frac{\theta}{2} + i\frac{\mu}{2} \right)}{\cosh \frac{\theta}{2} \cosh^2 \left(\frac{\theta}{2} - 2i\mu \right)} S(\theta)^2 \bar{S}(\theta + i\mu) \bar{S}(\theta - i\mu) \\ R(\lambda_1 = \lambda_2 = \pm e^{\pm i\mu}, z_1 = z_2 = e^{\mp i\mu}, \theta) \quad (35)$$

The overall factor which depends on $S(\theta)$ and $\bar{S}(\theta)$ will be fix by bootstrap. The fused S matrix satisfies the gYB equation for the circular graph defined by the subset of nodes, of the daisy graph, in this particular case three, living on the circle. The fusion procedure we have used able us to get rid of the unpleasant R factors which obscure the physics of the model.

ii) Bootstrap

Our next step will be to fix the overall factors by imposing crossing, unitarity and bootstrap on the fused scattering S - matrix. The geometry of the graph and the dependence of this overall factor on the undetermined functions $S(\theta)$ and $\bar{S}(\theta)$ suggest already the answer, namely the bootstrap factors of circular graphs. It is rather amusing and interesting to observe that the structure of the bootstrap factors $Z_{1,1} = Z_{2,2}$ for circular graphs (27) agrees with the one obtained in (35) by fusion, provided we make the following identification:

$$Z_{1,1}(\theta) = -3i \frac{\sinh \frac{\theta}{2} \cosh \left(\frac{\theta}{2} + \frac{i\pi}{6} \right) \cosh \left(\frac{\theta}{2} - \frac{i\pi}{6} \right)}{\cosh \frac{\theta}{2} \cosh^2 \left(\frac{\theta}{2} - \frac{2\pi i}{3} \right) \sinh \left(\frac{\theta}{2} - \frac{i\pi}{6} \right)} S(\theta)^2 S\left(\frac{2\pi i}{3} - \theta\right) S\left(\frac{4\pi i}{3} - \theta\right) \quad (36)$$

From the equation (27) we finally get the following expression of $S(\theta)$:

$$S(\theta) = C \frac{\cosh \left(\frac{\theta}{2} - \frac{2\pi i}{3} \right)}{\left(\sinh \frac{3\theta}{2} \right)^{1/2}} \prod_{j=1}^{\infty} \frac{\Gamma \left(\frac{\theta}{2\pi i} + j + \frac{1}{3} \right) \Gamma \left(\frac{\theta}{2\pi i} + j - \frac{1}{3} \right)}{\Gamma \left(-\frac{\theta}{2\pi i} + j + \frac{1}{3} \right) \Gamma \left(-\frac{\theta}{2\pi i} + j - \frac{1}{3} \right)} \quad (37)$$

where C is some constant. Strictely speaking eq.(37) is formal since the infinite product is actually divergent!. This divergency actually cancels out in equation (36) when reproducing $Z_{1,1}$ from $S(\theta)$. All these considerations provide a certain amount of evidence for interpreting the radial solitons as elementary constituents and the physical particles of daisy graph models as composite solitons. With respect to this second point we should make the following remark. From relations (37) we get the explicit value of the function $S(\theta)$, this function have a zero at the value $\theta = \frac{i\pi}{3}$ (not taking into account the divergence mention above) and therefore the composite state $(a \pm 1, a)$ cannot be taken "sensu stricto" as a bound state of two radial solitons. This can be interpreted as reflecting some kind of confinement of the radial solitons i.e. that they cannot appear as real asymptotic particles.

To finish this section we would like to make a comment on the possible scattering theory for the $D_{k+2}(t_2)$ model. Fusing the tentative solution conjectured in equation

(23) we can obtain the scattering matrices for a confined spectrum containing now two different types of composite solitons namely $(a, *_\alpha, b)$ for $\alpha = 1, 2$. These S matrices are the product of a $N = 0$ part obtained by fusing the Ising components and the same $N = 2$ part that for the $A_{k+1}(t_2)$.

The physical picture that emerges from the previous analysis of the daisy graph scattering is quite intriguing and certainly deserves a deeper study.

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